Implement Tasks That Promote Reasoning and Problem Solving

Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.

Effective mathematics teaching uses tasks as one way to motivate student learning and help students build new mathematical knowledge through problems solving. Research on the use of mathematical tasks over the last two decades has yielded three major findings:

1. Not all tasks provide the same opportunities for student thinking and learning. (Hiebert et al. 1997; Stein et al. 2009)

2. Student learning is greatest in classrooms where the tasks consistently encourage high-level student thinking and reasoning and least in classrooms where the tasks are routinely procedural in nature. (Boaler and Staples 2008; Hiebert and Wearne 1993; Stein and Lane 1996)

3. Tasks with high cognitive demands are the most difficult to implement well and are often transformed into less demanding tasks during instruction. (Stein, Grover, and Henningsen 1996; Stigler and Hiebert 2004)

To ensure that students have the opportunity to engage in high-level thinking, teachers must regularly select and implement tasks that promote reasoning and problem solving. These tasks encourage reasoning and access to the mathematics through multiple entry points, including the use of different representations and tools, and they foster the solving of problems through varied solution strategies.

Furthermore, effective teachers understand how contexts, culture, conditions, and language can be used to create mathematical tasks that draw on students’ prior knowledge and experiences (Cross et al. 2012; Kisker et. al. 2012; Moschkovich 1999, 2011) or that offer students a common experience from which their work on mathematical tasks emerges (Boaler 1997; Dubinsky and Wilson 2013; Wager 2012). As a result of teachers’ efforts to incorporate these elements into mathematical tasks, students’ engagement in solving these tasks is more strongly connected with their sense of identity, leading to increased engagement and motivation in mathematics (Aguirre, Mayfield-Ingram, and Martin 2013; Boaler 1997; Hogan 2008; Middleton and Jansen 2011).

Discussion

Mathematical tasks can range from a set of routine exercises to a complex and challenging problem that focuses students’ attention on a particular mathematical idea. Stein and colleagues
(Stein, Grover, and Henningsen 1996; Stein and Smith 1998) have developed a taxonomy of mathematical tasks based on the kind and level of thinking required to solve them. Smith and Stein (1998) show the characteristics of higher- and lower-level tasks and provide samples in each category; figure 3 reproduces their list of the characteristics of tasks at four levels of cognitive demand, and figure 4 provides examples of tasks at each of the levels.

Levels of Demands

**Lower-level demands (memorization):**
- Involve either reproducing previously learned facts, rules, formulas, or definitions or committing facts, rules, formulas or definitions to memory
- Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure
- Are not ambiguous. Such tasks involve the exact reproduction of previously seen material, and what is to be reproduced is clearly and directly stated.
- Have no connection to the concepts or meaning that underlie the facts, rules, formulas, or definitions being learned or reproduced

**Lower-level demands (procedures without connections):**
- Are algorithmic. Use of the procedure either is specifically called for or is evident from prior instruction, experience, or placement of the task.
- Require limited cognitive demand for successful completion. Little ambiguity exists about what needs to be done and how to do it.
- Have no connection to the concepts or meaning that underlie the procedure being used
- Are focused on producing correct answers instead of on developing mathematical understanding
- Require no explanations or explanations that focus solely on describing the procedure that was used

**Higher-level demands (procedures with connections):**
- Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas
- Suggest explicitly or implicitly pathways to follow that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts
- Usually are represented in multiple ways, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations helps develop meaning.
- Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with conceptual ideas that underlie the procedures to complete the task successfully and that develop understanding.

**Higher-level demands (doing mathematics):**
- Require complex and nonalgorithmic thinking—a predictable, well-rehearsed approach or pathway is not explicitly suggested by the task, task instructions, or a worked-out example
- Require students to explore and understand the nature of mathematical concepts, processes, or relationships
- Demand self-monitoring or self-regulation of one's own cognitive processes
- Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task
- Require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions
- Require considerable cognitive effort and may involve some level of anxiety for the student because of the unpredictable nature of the solution process required

These characteristics are derived from the work of Doyle on academic tasks (1988) and Resnick on high-level-thinking skills (1987), the Professional Standards for Teaching Mathematics (NCTM 1991), and the examination and categorization of hundreds of tasks used in QUASAR classrooms (Stein, Grover, and Henningsen 1996; Stein, Lane, and Silver 1996).

Fig. 3. Characteristics of mathematical tasks at four levels of cognitive demand. From Smith and Stein (1998).
Fig. 4. Sample tasks for four levels of cognitive demand. From Smith and Stein (1998).

From the perspective of this taxonomy, mathematical tasks are viewed as placing higher-level cognitive demands on students when they allow students to engage in active inquiry and exploration or encourage students to use procedures in ways that are meaningfully connected with concepts or understanding. Tasks that encourage students to use procedures, formulas, or algorithms in ways that are not actively linked to meaning, or that consist primarily of memorization or the reproduction of previously memorized facts, are viewed as placing lower-level cognitive demands on students. Consider figure 5, which shows two tasks, both of which might be used in an algebra unit that includes analyzing and solving pairs of simultaneous equations.
<table>
<thead>
<tr>
<th>Task A: Smartphone Plans</th>
<th>Task B: Solving systems of equations</th>
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</table>
| You are trying to decide which of two smartphone plans would be better. Plan A charges a basic fee of $30 per month and 10 cents per text message. Plan B charges a basic fee of $50 per month and 5 cents per text message. How many text messages would you need to send per month for plan B to be the better option? Explain your decision. | Solve each of the following systems:  
-4x - 2y = -12  
4x + 8y = -24  
x - y = 11  
2x + y = 19  
8x + y = -1  
-3x + y = -5  
5x + y = 9  
10x - 7y = -18 |

(Adapted from Illustrative Mathematics Illustrations: www.illustrativemathematics.org/illustrations/469.)

Fig. 5. Comparison of tasks with different cognitive demand

Task A is a high-level task, since no specific pathway has been suggested or is implied, and students could use several different approaches to enter and solve the task (e.g., guess and check, make a table, graph equations to find the point of intersection, solve a system of two linear equations by using algebra). Further, students must put forth effort to determine and enact a course of action and justify the reasonableness and accuracy of their solutions. By contrast, task B is a low-level task because it is likely that students are expected to use a specific memorized procedure that leaves little or no ambiguity about what they need to do. The mathematics that students can learn in doing a high-level task is significantly different from the mathematics that they learn from low-level tasks. Over time, the cumulative effect of the use of mathematics tasks is students’ implicit development of ideas about the nature of mathematics—about whether mathematics is something that they personally can make sense of and how long and how hard they should have to work to solve any mathematical task.

It is important to note that not all tasks that promote reasoning and problem solving have to be set in a context or need to consume an entire class period or multiple days. What is critical is that a task provide students with the opportunity to engage actively in reasoning, sense making, and problem solving so that they develop a deep understanding of mathematics. Take, for example, the task on exponential functions in figure 6, which calls on students to analyze functions by using visual representations. In working on this task, students explore what happens to the graph of the function when the values of \( a \) change, and through their use of representations, they generalize the behavior of the function.
Using your graphing calculator, investigate the changes that occur in the graph of \( y = a^x \) for different values of \( a \), where \( a \) is any real number. Explain what happens in the following cases:

1. \( a > 1 \)
2. \( a = 1 \)
3. \( 0 < a < 1 \)
4. \( a = 0 \)
5. \( a < 0 \)

Fig. 6. An algebra task requiring students to use graphical representations to analyze exponential functions

This task promotes problem solving because students are positioned to explore the situation without being told in advance what to expect. Through reasoning about this task, they are likely to determine the general shape of the graph of the function (e.g., when \( a > 1 \), the graph starts out “flat” and close to the \( x \)-axis and then shoots up; when \( 0 < a < 1 \), the graph shows a rapidly shrinking function), what occurs at 0 and 1, and the difference between a growth function and a decay function. Extending this discussion to the case of \( a < 0 \) provides an important opportunity for students to learn why exponential functions are restricted to \( a \geq 0 \).

Tasks engaging students in reasoning and problem solving are not limited to middle and high school content. Consider the task in figure 7, in which students in kindergarten–grade 1 decompose the number 10 into pairs in more than one way.

There are 10 cars in the parking lot. Some of the cars are red and some of the cars are black. How many red cars and how many black cars could be in the parking lot?

Think of as many different combinations of cars as you can.

Show your solutions in as many ways as you can with cubes, drawings, or words, and write an equation for each solution.

Fig. 7. A task for K–grade 1 on number pairs that make 10. Adapted from the North Carolina Department of Public Instruction; http://commoncoretasks.ncdpi.wikispaces.net/First+Grade+Tasks.
In this problem, students identify one or more combinations that equal 10, using drawings, cubes, or other tools (e.g., fingers, ten frame, Rekenrek) as needed to support their problem solving and explaining. This is a high-level task for most kindergarten and first-grade students because they have not yet learned these combinations, and they can use a variety of strategies (e.g., trial and error, counting up to 10 from a selected number, decomposing 10 into two sets) to determine the combinations that will work. Through the process of solving this task, students may recognize similar combinations (e.g., $4 + 6 = 6 + 4$) and begin to see number patterns (e.g., $1 + 9, 2 + 8, 3 + 7$; as one number gets bigger by 1, the other number gets smaller by 1).

In determining the level of task, it is important to consider the prior knowledge and experiences of the students who will be engaged in the task. Tasks may begin as high-level tasks for students who are initially learning about the underlying mathematics (e.g., systems of linear equations, behaviors of functions, number combinations). Eventually, as students solidify their understanding of the underlying mathematics, these tasks may become more routine experiences for them. Students then need tasks that further extend these mathematical ideas in ways that continue to deepen understanding and strengthen mathematical reasoning and problem solving.

**Illustration**

Although selecting tasks that promote reasoning and problem solving is a critical first step, giving the task to students does not guarantee that students will actually engage in the task at a high level. Consider the comparison that figure 8 presents in the implementation of task A, Smartphone Plans, shown in figure 5, in two algebra classrooms.

Note that although Ms. Carson uses a task that could promote reasoning, as soon as she sees students struggling, she provides them with a pathway for solving the task. By taking over the thinking for her students, Ms. Carson removes their opportunity to engage deeply and meaningfully with the mathematics and leaves them simply to apply a specific procedure.

By contrast, when Ms. McDonald sees her students struggling to figure out what to do, she provides suggestions that will help them make progress on the task without giving them a specific pathway to follow. This is the approach that NCTM (2000 p. 19) has long advocated:

> Teachers must decide what aspects of a task to highlight, how to organize and orchestrate the work of the students, what questions to ask to challenge those with varied levels of expertise, and how to support students without taking over the process of thinking for them and thus eliminating the challenge.

As a result of the way in which Ms. McDonald orchestrates the lesson, students have the opportunity to consider different strategies and engage in mathematical problem solving at a
As students in two algebra classrooms begin working with their partners on the Smartphone Plans task, it becomes evident that students are struggling to get started.

**Ms. Carson’s classroom**

Ms. Carson calls the class together and tells the students that they first need to write equations for both smartphone plans. She writes \( y = mx + b \) on the board and asks students what \( m \) and \( b \) would be for each phone plan. Once they have established the two equations, she goes to the board and creates a table that contains three columns: \( x \) (number of text messages), \( y_1 \) (cost for plan A), and \( y_2 \) (cost for plan B). She suggests that they begin with 0 text messages and then increment the \( x \) values in the table by 10. The students resume their work with their partners and easily complete the table, identifying (400, 70) as the point of intersection of the two equations.

**Ms. McDonald’s classroom**

Ms. McDonald poses questions to students as she walks around the room. When she notices students struggling to get started, she asks them how much it will cost to send one text message in each plan. This question enables her to be sure that the students understand the relationship among the number of messages, the cost per message, and the basic fee. She asks them which plan will cost more for a specific number of messages and to consider whether this plan’s cost will always be higher. Then she leaves the partners to discuss ways to use this information to solve the problem. As the students continue working, she observes different approaches, hears debates on whether the answer is 400 messages or 401 messages, and plans how to sequence the whole-class discussion to analyze and compare the varied strategies.

**Fig. 8. A look inside two algebra classrooms at the implementation of Smartphone Plans (task A in fig. 5)**

high level of cognitive demand. Moreover, and most important, the students are challenged to deepen their understanding of linear equations and what the point of intersection means, both graphically and contextually.

**Teacher and student actions**

For students to learn mathematics with understanding, they must have opportunities to engage on a regular basis with tasks that focus on reasoning and problem solving and make possible multiple entry points and varied solution strategies. The actions listed in the table on the next page provide a summary of what teachers and students need to do when implementing such tasks in the mathematics classroom. It is important to note that tasks that focus on learning and applying procedures do have a place in the curriculum and are necessary for developing fluency. Such tasks, however, should not dominate instruction and preempt the use of tasks that promote reasoning. Instead, these tasks should build on and emerge from these sense-making and problem-solving experiences.
## Implement tasks that promote reasoning and problem solving
Teacher and student actions

<table>
<thead>
<tr>
<th>What are teachers doing?</th>
<th>What are students doing?</th>
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</thead>
<tbody>
<tr>
<td>Motivating students’ learning of mathematics through opportunities for exploring and solving problems that build on and extend their current mathematical understanding.</td>
<td>Persevering in exploring and reasoning through tasks.</td>
</tr>
<tr>
<td>Selecting tasks that provide multiple entry points through the use of varied tools and representations.</td>
<td>Taking responsibility for making sense of tasks by drawing on and making connections with their prior understanding and ideas.</td>
</tr>
<tr>
<td>Posing tasks on a regular basis that require a high level of cognitive demand.</td>
<td>Using tools and representations as needed to support their thinking and problem solving.</td>
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<tr>
<td>Supporting students in exploring tasks without taking over student thinking.</td>
<td>Accepting and expecting that their classmates will use a variety of solution approaches and that they will discuss and justify their strategies to one another.</td>
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<tr>
<td>Encouraging students to use varied approaches and strategies to make sense of and solve tasks.</td>
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