Standards for Mathematical Practice:
Standard 1: Make sense of problems and persevere in solving them

The Standard:
Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem.

Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

Classroom Observations:
Teachers who are developing students’ capacity to “make sense of problems and persevere in solving them” develop ways of framing mathematical challenges that are clear and explicit, and then check in repeatedly with students to help them clarify their thinking and their process. An early childhood teacher might ask her students to work in pairs to evaluate their approach to a problem, telling a partner to describe their process, saying “what [they] did, and what [they] might do next time.” A middle childhood teacher might post a set of different approaches to a solution, asking students to identify “what this mathematician was thinking or trying out” and evaluating the success of the strategy. An early adolescence teacher might have students articulate a specific way of laying out the terrain of a problem and evaluating different starting points for solving. A teacher of adolescents and young adults might frame the task as a real-world design conundrum, inviting students to engage in a “tinkering” process of working toward mathematical proof, changing course as necessary as they develop their thinking. Visit the video excerpts at Inside Mathematics website http://www.insidemathematics.org/index.php/mathematical-practice-standards to view multiple examples of teachers engaging students in sense making and mathematical perseverance.

Students:
- Analyze and explain the meaning of the problem
- Actively engage in problem solving (Develop, carry out, and refine a plan)
- Show patience and positive attitudes
- Ask if their answers make sense
- Check their answers with a different method

Because Teachers:
- Pose rich problems and/or ask open ended questions
- Provide wait-time for processing/finding solutions
- Circulate to pose probing questions and monitor student progress
- Provide opportunities and time for cooperative problem solving and reciprocal teaching

Math Practice | Key Points | Students might think or do:
--- | --- | ---
Make sense of problems and persevere in solving them | • explain to themselves the meaning of a problem  
• look for entry points to its solution  
• analyze givens, constraints, relationships, and goals | • “I tried that approach to solving the problem and it didn’t work. What’s another way I can try to solve it?”  
• “What’s a useful way to begin working on this problem?” |
### Math Practice

#### Key Points

- Make sense of problems and persevere in solving them (cont.)

#### Students might think or do.

- They can set up a series of steps to follow to get themselves to the answer.
- “There’s another problem I’ve done that’s like this that might help me here.”
- “This isn’t working; I need to try something else.”

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#### Math Practice Key Points

Students might think or do.

<table>
<thead>
<tr>
<th>Make Sense of Problems and Persevere in Solving Them</th>
<th>Questions to Develop Mathematical Thinking</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Interpret and make meaning of the problem to find a starting point. Analyze what is given in order to explain to themselves the meaning of the problem.</td>
<td>How would you describe the problem in your own words?</td>
</tr>
<tr>
<td>• Plan a solution pathway instead of jumping to a solution.</td>
<td>How would you describe what you are trying to find?</td>
</tr>
<tr>
<td>• Monitor their progress and changed their approach if necessary.</td>
<td>What do you notice about…?</td>
</tr>
<tr>
<td>• See relationships between various representations.</td>
<td>What information is given in the problem?</td>
</tr>
<tr>
<td>• Relate current situations to concepts or skills previously learned and connect mathematical ideas to one another.</td>
<td>Describe the relationship between the quantities.</td>
</tr>
<tr>
<td>• Continually ask themselves, “Does this make sense?” Can understand various approaches to solutions.</td>
<td>Describe what you have already tried.</td>
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<tr>
<td></td>
<td>What might you change?</td>
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<td></td>
<td>Talk me through the steps you’ve used to this point.</td>
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<tr>
<td></td>
<td>What steps in the process are you most confident about?</td>
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<td></td>
<td>What are some other strategies you might try?</td>
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<tr>
<td></td>
<td>What are some other problems that are similar to this one?</td>
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<tr>
<td></td>
<td>How might you use one of your previous problems to help you begin?</td>
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<td></td>
<td>How else might you organize…represent…show…?</td>
</tr>
</tbody>
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#### Practice Needs Improvement

**Task:**
- Is strictly procedural.
- Does not require students to check solutions for errors.

**Teacher:**
- Does not allow for wait time; asks leading questions to rush through task.
- Does not encourage students to individually process the tasks.
- Is focused solely on answers rather than processes and reasoning..

#### Emerging (teacher does thinking)

**Task:**
- Is overly scaffolded or procedurally “obvious”.
- Requires students to check answers by plugging in numbers.

**Teacher:**
- Allots too much or too little time for the task.
- Encourages students to individually complete the tasks, but does not ask them to evaluate the processes they used.
- Explains the reasons behind procedural steps.
- Does not check errors publicly.

#### Proficient (teacher mostly models)

**Task:**
- Is cognitively demanding.
- Has more than one entry point.
- Requires a balance of procedural fluency and conceptual understanding.
- Requires students to check solutions for errors using one other solution path.

**Teacher:**
- Allows ample time for all students to struggle with the task.
- Expects students to evaluate processes implicitly.
- Models making sense of the task (given situation) and the proposed solution.

#### Exemplary (students take ownership)

**Task:**
- Allows for multiple entry points and solution paths.
- Requires students to defend and justify their solution by comparing multiple solution paths..

**Teacher:**
- Differentiates to keep advanced students challenged during work time.
- Integrates time for explicit meta-cognition.
- Expects students to make sense of task and proposed solution.
Standards for Mathematical Practice:
Standard 2: Reason abstractly and quantitatively

The Standard:
Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

Classroom Observations:
Teachers who are developing students’ capacity to "reason abstractly and quantitatively" help their learners understand the relationships between problem scenarios and mathematical representation, as well as how the symbols represent strategies for solution. A middle childhood teacher might ask her students to reflect on what each number in a fraction represents as parts of a whole. A different middle childhood teacher might ask his students to discuss different sample operational strategies for a patterning problem, evaluating which is the most efficient and accurate means of finding a solution. Visit the video excerpts below to view these teachers engaging their students in abstract and quantitative reasoning.


<table>
<thead>
<tr>
<th>Students:</th>
<th>Because Teachers:</th>
</tr>
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<tbody>
<tr>
<td>• Represent a problem with symbols</td>
<td>• Ask students to explain their thinking regardless of accuracy</td>
</tr>
<tr>
<td>• Explain their thinking</td>
<td>• Highlight flexible use of numbers</td>
</tr>
<tr>
<td>• Use numbers flexibly by applying properties of operations and place value</td>
<td>• Facilitate discussion through guided questions and representations</td>
</tr>
<tr>
<td>• Examine the reasonableness of their answers/calculations</td>
<td>• Accept varied solutions/representations</td>
</tr>
</tbody>
</table>

Math Solutions

<table>
<thead>
<tr>
<th>Math Practice</th>
<th>Key Points</th>
<th>Students might think or do:</th>
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</thead>
<tbody>
<tr>
<td>Reason Abstractly and Quantitatively</td>
<td>• make sense of quantities and their relationships in problem situations</td>
<td>• “How can I capture important information in a diagram or model?”</td>
</tr>
<tr>
<td></td>
<td>• decontextualize - abstract and represent a problem situation symbolically and manipulate those symbols without attending to their referents</td>
<td>• “What solution path does this diagram or model imply?”</td>
</tr>
<tr>
<td></td>
<td>• contextualize - pause during problem solving to connect symbolic work back to the context of the problem</td>
<td>• “OK, I’ve done all these calculations; now, what does that mean in the problem? Does my answer make sense for answering this problem?”</td>
</tr>
<tr>
<td>Reason Abstractly and Quantitatively (cont.)</td>
<td>Questions to Develop Mathematical Thinking</td>
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<td>------------------------------------------</td>
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</tbody>
</table>
| • Pay attention to the important quantities and relationships between them  
  • use representations to highlight those relationships and the underlying mathematical structure of a problem | What do the numbers used in the problem represent?  
What is the relationship of the quantities?  
How is ___ related to ___?  
What properties might we use to find a solution?  
What does ___ mean to you? (e.g., symbol, quantity, diagram)  
How did you decide in this task that you needed to use . . . ?  
Could we have used another operation or property to solve this task?  
Why or why not? |
| • Given the problem: There are 3/5 as many boys as girls. If there are 45 boys, how many girls are there?, a student can create a diagram that shows the relationship between the number. |  
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<table>
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<tr>
<th>Practice</th>
<th>Needs Improvement</th>
<th>Emerging (teacher does thinking)</th>
<th>Proficient (teacher mostly models)</th>
<th>Exemplary (students take ownership)</th>
</tr>
</thead>
</table>
| Reason Abstractly and Quantitatively | Task:  
◊ Lacks context.  
◊ Does not make use of multiple representations or solution paths.  
Teacher:  
◊ Does not expect students to interpret representations.  
◊ Expects students to memorize procedures with no connection to meaning. | Task:  
◊ Is embedded in a contrived context.  
Teacher:  
◊ Expects students to model and interpret tasks using a single representation.  
◊ Explains connections between procedures and meaning. | Task:  
◊ Has realistic context.  
◊ Requires students to frame solutions in a context.  
◊ Has solutions that can be expressed with multiple representations.  
Teacher:  
◊ Expects students to interpret and model using multiple representations.  
◊ Provides structure for students to connect algebraic procedures to contextual meaning.  
◊ Links mathematical solution with a question’s answer. | Task:  
◊ Has relevant realistic context.  
Teacher:  
◊ Expects students to interpret, model, and connect multiple representations.  
◊ Prompts students to articulate connections between algebraic procedures and contextual meaning. |

CCSS-M Flip Books: [http://katm.org/wp/common-core/]
Standards for Mathematical Practice:
Standard 3: Construct Viable Arguments and Critique the Reasoning of Others

The Standard:
Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

Classroom Observations:
Teachers who are developing students’ capacity to "construct viable arguments and critique the reasoning of others" require their students to engage in active mathematical discourse. This might involve having students explain and discuss their thinking processes aloud, or signaling agreement/disagreement with a hand signal. A middle childhood teacher might post multiple approaches to a problem and ask students to identify plausible rationales for each approach as well as any mistakes made by the mathematician. An early adolescence teacher might post a chart showing a cost-analysis comparison of multiple DVD rental plans and ask his students to formulate and defend a way of showing when each plan becomes most economical. A teacher of adolescents and young adults might actively engage her students in extended conjecture about conditions for proof in the construction of quadrilaterals, testing their assumptions and questioning their approaches. Visit the video excerpts at Inside Mathematics website: http://www.insidemathematics.org/index.php/mathematical-practice-standards to view multiple examples of teachers engaging students in formulating, critiquing and defending arguments of mathematical reasoning.

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<th>Students:</th>
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<tbody>
<tr>
<td>Make reasonable guesses to explore their ideas</td>
<td>Provide opportunities for students to listen to or read the conclusions and arguments of others</td>
</tr>
<tr>
<td>Justify solutions and approaches</td>
<td>Establish and facilitate a safe environment for discussion</td>
</tr>
<tr>
<td>Listen to the reasoning of others, compare arguments, and decide if the arguments of others make sense</td>
<td>Ask clarifying and probing questions</td>
</tr>
<tr>
<td>Ask clarifying and probing questions</td>
<td>Avoid giving too much assistance (e.g., providing answers or procedures)</td>
</tr>
</tbody>
</table>

Math Solutions
<table>
<thead>
<tr>
<th>Math Practice</th>
<th>Key Points</th>
<th>Students might think or do:</th>
</tr>
</thead>
</table>
| **Construct viable arguments and critique the reasoning of others** | • make conjectures and build a logical progression of statements to explore the truth of their conjectures  
• analyze situations by breaking them into cases  
• recognize and use counterexamples  
• justify conclusions, communicate them to others, and respond to the arguments of others  
• distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is | • A student can state a rule for a pattern, and can explain why their rule works for that pattern.  
• When someone claims “multiplying two numbers gives you an answer bigger than either the numbers,” a student can think about: - what happens when you multiply 2 whole numbers; - what happens when you multiply by a fraction; - what happens when you multiply 2 fractions |

**MP 3**

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**Construct viable arguments and critique the reasoning of others.**

- Analyze problems and use stated mathematical assumptions, definitions, and established results in constructing arguments.
- Justify conclusions with mathematical ideas.
- Listen to the arguments of others and ask useful questions to determine if an argument makes sense.
- Ask clarifying questions or suggest ideas to improve/revise the argument.
- Compare two arguments and determine correct or flawed logic.

**Questions to Develop Mathematical Thinking**

- What mathematical evidence would support your solution?  
- How can we be sure that...?  
- How could you prove that...?  
- Will it still work if...?  
- What were you considering when...?  
- How did you decide to try that strategy?  
- How did you test whether your approach worked?  
- How did you decide what the problem was asking you to find? (What was unknown?)  
- Did you try a method that did not work?  
- Why didn’t it work?  
- Would it ever work?  
- Why or why not?  
- What is the same and what is different about...?  
- How could you demonstrate a counter-example?


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</tr>
</thead>
</table>
| **Construct Viable Arguments and Critique Reasoning of Others** | Task:  
◊ Is ambiguously stated.  
Teacher:  
◊ Does not ask students to present arguments or solutions.  
◊ Expects students to follow a given solution path without opportunities to make conjectures. | Task:  
◊ Is not at the appropriate level.  
Teacher:  
◊ Does not help students differentiate between assumptions and logical conjectures.  
◊ Asks students to present arguments but not to evaluate them.  
◊ Allows students to make conjectures without justification. | Task:  
◊ Avoids single steps or routine algorithms.  
Teacher:  
◊ Identifies students’ assumptions.  
◊ Models evaluation of student arguments.  
◊ Asks students to explain their conjectures. | Teacher:  
◊ Helps students differentiate between assumptions and logical conjectures.  
◊ Prompts students to evaluate peer arguments.  
◊ Expects students to formally justify the validity of their conjectures. |

*Institute for Advanced Study*  
*Park City Mathematics Institute*
Standards for Mathematical Practice:
Standard 4: Model With Mathematics

The Standard:
Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

Classroom Observations:
Teachers who are developing students’ capacity to "model with mathematics" move explicitly between real-world scenarios and mathematical representations of those scenarios. A middle childhood teacher might pose a scenario of candy boxes containing multiple flavors to help students identify proportions and ratios of flavors and ingredients. An early adolescence teacher might represent a comparison of different DVD rental plans using a table, asking the students whether or not the table helps directly compare the plans or whether elements of the comparison are omitted. A teacher of adolescents and young adults might pose a "kite factory" scenario, in which advanced students are asked to determine the conditions for always creating a particular shape of kite given the dimensions of the diagonals and the angle of intersection. Visit the video excerpts at Inside Mathematics website: http://www.insidemathematics.org/index.php/mathematical-practice-standards to view multiple examples of teachers engaging students in mathematical modeling.

<table>
<thead>
<tr>
<th>Students:</th>
<th>Because Teachers:</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Apply prior knowledge to new problems and reflect</td>
<td>• Pose problems connected to previous concepts</td>
</tr>
<tr>
<td>• Use representations to solve real life problems</td>
<td>• Provide a variety of real world contexts</td>
</tr>
<tr>
<td>• Apply formulas and equations where appropriate</td>
<td>• Use intentional representations</td>
</tr>
</tbody>
</table>

Math Solutions
<table>
<thead>
<tr>
<th>Math Practice</th>
<th>Key Points</th>
<th>Students might think or do:</th>
</tr>
</thead>
</table>
| Model With Mathematics | • make assumptions and approximations to simplify a complicated situation, realizing that these may need revision later  
• identify important quantities in a practical situation  
• map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas  
• analyze those relationships mathematically to draw conclusions  
• interpret their mathematical results in the context of the situation  

*MP 4* | • A student can state a rule for a pattern, and can explain why their rule works for that pattern.  
• When someone claims “multiplying two numbers gives you an answer bigger than either the numbers,” a student can think about: - what happens when you multiply 2 whole numbers; - what happens when you multiply by a fraction; - what happens when you multiply 2 fractions  

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</table>
| Model With Mathematics | Task:  
◊ Requires students to identify variables and to perform necessary computations.  

Teacher:  
◊ Identifies appropriate variables and procedures for students.  
◊ Does not discuss appropriateness of models.  

Teacher:  
◊ Identifies appropriate variables and procedures.  
◊ Does not discuss appropriateness of model. | Task:  
◊ Requires students to identify variables and to compute and interpret results.  

Teacher:  
◊ Verifies that students have identified appropriate variables and procedures.  
◊ Explains the appropriateness of model. | Task:  
◊ Requires students to identify variables, compute results, and report findings using a mixture of representations.  
◊ Illustrates the relevance of the mathematics involved.  
◊ Requires students to identify extraneous or missing information.  

Teacher:  
◊ Asks questions to help students identify appropriate variables and procedures.  
◊ Facilitates discussions in evaluating the appropriateness of model. | Task:  
◊ Requires students to identify variables, compute and interpret results, and justify the reasonableness of their results and procedures.  

Teacher:  
◊ Expects students to justify their choice of variables and procedures.  
◊ Gives students opportunity to evaluate the appropriateness of model. |
Standards for Mathematical Practice:

Standard 5. Use Appropriate Tools Strategically

The Standard:
Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

Classroom Observations:
Teachers who are developing students’ capacity to "use appropriate tools strategically" make clear to students why the use of manipulatives, rulers, compasses, protractors, and other tools will aid their problem solving processes. A middle childhood teacher might have his students select different color tiles to show repetition in a patterning task. A teacher of adolescents and young adults might have established norms for accessing tools during the students' group "tinkering processes," allowing students to use paper strips, brass fasteners, and protractors to create and test quadrilateral "kite" models. Visit the video excerpts at Inside Mathematics website: http://www.insidemathematics.org/index.php/mathematical-practice-standards to view multiple examples of teaching students to use appropriate tools strategically.

Students:
- Select and use tools strategically (and flexibly) to visualize, explore, and compare information
- Use technological tools and resources to solve problems and deepen understanding

Because Teachers:
- Make appropriate tools available for learning (calculators, concrete models, digital resources, pencil/paper, compass, protractor, etc.)
- Use tools with their instruction

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| Use Appropriate Tools Strategically | - Are familiar with tools appropriate for their grade or course and can make sound decisions about when each of these tools might be helpful.  
- Identify relevant external mathematical resources, such as digital content located on a website, and to use them to pose or solve problems. | - A student wants to see how the difference between values in a table changes, so she begins by making a table, then decides to put the information in a spreadsheet to more easily do the calculations, and draw conclusions from the results. 
- A student is having trouble visualizing a situation with a geometric shape, so he creates it in a geometry software application and is able to move the shape around to see how some parts of the shape change while keeping certain characteristics of the shape the same. |
### Use Appropriate Tools Strategically  

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</table>
| Use Appropriate Tools Strategically | Task: ◊ Does not incorporate additional learning tools.  
Teacher: ◊ Does not incorporate additional learning tools. | Task: ◊ Lends itself to one learning tools.  
◊ Does not involve mental computations or estimation.  
Teacher: ◊ Demonstrates use of appropriate learning tool. | Task: ◊ Lends itself to multiple learning tools.  
◊ Gives students opportunity to develop fluency in mental computations.  
Teacher: ◊ Choose appropriate learning tools for student use.  
◊ Models error checking by estimation. | Task: ◊ Requires multiple learning tools (i.e., graph paper, calculator, manipulatives)  
◊ Requires students to demonstrate fluency in mental computations.  
Teacher: ◊ Allows students to choose learning tools.  
◊ Creatively finds appropriate alternatives where tools are not available. |

### Questions to Develop Mathematical Thinking

- What mathematical tools could we use to visualize and represent the situation?
- What information do you have?
- What do you know that is not stated in the problem?
- What approach are you considering trying first?
- What estimate did you make for the solution?
- In this situation would it be helpful to use...a graph..., number line..., ruler..., diagram..., calculator..., manipulative?
- Why was it helpful to use...?
- What can using a _____ show us, that _____ may not?
- In what situations might it be more informative or helpful to use...?

Standards for Mathematical Practice:
Standard 6. Attend to Precision

The Standard:
Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

Classroom Observations:
Teachers who are developing students’ capacity to "attend to precision" focus on clarity and accuracy of process and outcome in problem solving. A middle childhood teacher might engage his students in a "number talk" in which students use an in/out table and a plotted graph to "guess [the teacher’s] number." An early adolescence teacher might distribute cards with different symbol strings to his students, asking them to mingle to group and categorize their symbol strings, explaining and defending their groupings. A teacher of adolescents and young adults might continually probe her students to defend whether their requirements for a particular quadrilateral will always be the case, or whether there are some flaws in their group’s thinking that they need to refine and correct. Visit the video excerpts at Inside Mathematics website: [http://www.insidemathematics.org/index.php/mathematical-practice-standards](http://www.insidemathematics.org/index.php/mathematical-practice-standards) to view multiple examples of teachers engaging students in attending to precision.

<table>
<thead>
<tr>
<th>Students:</th>
<th>Because Teachers:</th>
</tr>
</thead>
</table>
| • Calculate accurately and efficiently  
• Explain their thinking using mathematics vocabulary  
• Use appropriate symbols and specify units of measure | • Recognize and model efficient strategies for computation  
• Use (and challenge students to use) mathematics vocabulary precisely and consistently |

**Math Practice**  
**Key Points**  
• Use clear definitions in discussion with others and in their own reasoning.  
• State the meaning of the symbols they choose, including using the equal sign consistently and appropriately.  
• Are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem.  
• Express numerical answers with a degree of precision appropriate for the problem context.  

**Students might think or do:**  
• A student rewrites his explanation to a problem using appropriate mathematics vocabulary.  
• A student learns why it is incorrect to write $14 + 4 = 18 + 5 = 23 \times 2 = 46$  
• “My calculator says 3.581279, but since I’m asked to find the number of inches, that’s not a number that makes sense to write for a measurement in inches. I’ll say 3.5” or 3.6”.

*Education Development Center, Inc.*
### Attend to Precision  
**MP6**

<table>
<thead>
<tr>
<th>Practice</th>
<th>Needs Improvement</th>
<th>Emerging (teacher does thinking)</th>
<th>Proficient (teacher mostly models)</th>
<th>Exemplary (students take ownership)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Attend to Precision</strong></td>
<td></td>
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</tr>
</tbody>
</table>
**Task:**  
◊ Gives imprecise instructions.  
**Teacher:**  
◊ Does not intervene when students are being imprecise  
◊ Does not point out when students fail to address the question completely or directly.  
| **Task:**  
◊ Has overly detailed or wordy instructions.  
**Teacher:**  
◊ Inconsistently intervenes when students are imprecise.  
◊ Identifies incomplete responses but does not require student to formulate further response.  
| **Task:**  
◊ Has precise instructions.  
**Teacher:**  
◊ Consistently demands precision in communication and in mathematical solutions.  
◊ Identifies incomplete responses and asks student to revise their response.  
| **Task:**  
◊ Includes assessment criteria for communication of ideas.  
**Teacher:**  
◊ Demands and models precision in communication and in mathematical solutions.  
◊ Encourages student to identify when others are not addressing the question completely.  

### Questions to Develop Mathematical Thinking

- What mathematical terms apply in this situation?
- How did you know your solution was reasonable?
- Explain how you might show that your solution answers the problem.
- Is there a more efficient strategy?
- How are you showing the meaning of the quantities?
- What symbols or mathematical notations are important in this problem?
- What mathematical language..., definitions..., properties can you use to explain?
- How could you test your solution to see if it answers the problem?


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Institute for Advanced Study  
Park City Mathematics Institute
Standards for Mathematical Practice:
Standard 7. Look For and Make Use of Structure

The Standard:
Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see \(7 \times 8\) equals the well remembered \(7 \times 5 + 7 \times 3\), in preparation for learning about the distributive property. In the expression \(x^2 + 9x + 14\), older students can see the 14 as \(2 \times 7\) and the 9 as \(2 + 7\). They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see \(5 - 3(x - y)^2\) as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers \(x\) and \(y\).

Classroom Observations:
Teachers who are developing students' capacity to "look for and make use of structure" help learners identify and evaluate efficient strategies for solution. An early childhood teacher might help students identify why using "counting on" is preferable to counting each addend by one, or why multiplication or division can be preferable to repeated addition or subtraction. A middle childhood teacher might help his students discern patterns in a function table to "guess my rule." A teacher of adolescents and young adults might focus on exploring geometric processes through patterns and proof. Visit the video excerpts at Inside Mathematics website: [http://www.insidemathematics.org/index.php/mathematical-practice-standards](http://www.insidemathematics.org/index.php/mathematical-practice-standards) to view multiple examples of teachers engaging students in identifying and making use of mathematical structure.

<table>
<thead>
<tr>
<th>Students:</th>
<th>Because Teachers:</th>
</tr>
</thead>
<tbody>
<tr>
<td>✗ Look for, develop, and generalize relationships and patterns</td>
<td>✗ Provide time for applying and discussing properties</td>
</tr>
<tr>
<td>✗ Apply reasonable thoughts about patterns and properties to new situations</td>
<td>✗ Ask questions about the application of patterns</td>
</tr>
<tr>
<td>✗ Highlight different approaches for solving problems</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Math Practice</th>
<th>Key Points</th>
<th>Students might think or do:</th>
</tr>
</thead>
</table>
| Look For and Make Use of Structure | • Look for similar mathematical structures across seemingly different problems  
• Use those similarities to help them reason about how to solve a problem  
• Can step back for an overview and shift perspective  
• Can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. | • “Since (problem A) and (problem B) are structurally the same, what do I know about solving (prob A) that will me think about solving (prob B)’?”  
• “Figuring out what to do with \(3(x + 2)\) is just like the work I did in (?th) grade when I learned that \(7 x 8\) is the same as \((7 x 3) + (7 x 5)\).”  
• Would recognize that \(17 + 2 (x + 1)\) will be odd because \(2 (x + 1)\) is even (since it’s 2 times some number) and 17 is odd, and an odd amount + an even amount will be odd.  
• Noticing that all numbers that have a remainder of 4 when divided by 5 will end in either 4 or 9.  
• “I recognize that \(1/3 (A + B + C)\) is really just the same as finding the average of 3 numbers.” |
<table>
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<th>Look For and Make Use of Structure</th>
<th>MP7</th>
<th>Questions to Develop Mathematical Thinking</th>
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<tbody>
<tr>
<td>• Apply general mathematical rules to specific situations.</td>
<td></td>
<td>What observations do you make about...?</td>
</tr>
<tr>
<td>• Look for the overall structure and patterns in mathematics.</td>
<td></td>
<td>What do you notice when...?</td>
</tr>
<tr>
<td>• See complicated things as single objects or as being composed of several objects.</td>
<td></td>
<td>What parts of the problem might you eliminate..., simplify...?</td>
</tr>
<tr>
<td><strong>Teacher:</strong></td>
<td></td>
<td>What patterns do you find in...?</td>
</tr>
<tr>
<td>◊ Requires students to automatically apply an algorithm to a task without evaluating its appropriateness.</td>
<td></td>
<td>How do you know if something is a pattern?</td>
</tr>
<tr>
<td>◊ Does not recognize students for developing efficient approaches to the task.</td>
<td></td>
<td>What ideas that we have learned before were useful in solving this problem?</td>
</tr>
<tr>
<td>◊ Requires students to apply the same algorithm to a task although there may be other approaches.</td>
<td></td>
<td>What are some other problems that are similar to this one?</td>
</tr>
<tr>
<td><strong>Teacher:</strong></td>
<td></td>
<td>How does this relate to...?</td>
</tr>
<tr>
<td>◊ Identifies individual students’ efficient approaches, but does not expand understanding to the rest of the class.</td>
<td></td>
<td>In what ways does this problem connect to other mathematical concepts?</td>
</tr>
<tr>
<td>◊ Demonstrates the same algorithm to all related tasks although there may be other more effective approaches.</td>
<td></td>
<td><strong>CCSS-M Flip Books:</strong> <a href="http://katm.org/wp/common-core/">http://katm.org/wp/common-core/</a></td>
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### Practice

<table>
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<tbody>
<tr>
<td><strong>Task:</strong></td>
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<td><strong>Teacher:</strong></td>
</tr>
<tr>
<td>◊ Prompts students to identify mathematical structure of the task in order to identify the most effective solution path.</td>
</tr>
<tr>
<td>◊ Encourages students about their choice of algorithm or solution path.</td>
</tr>
<tr>
<td><strong>Teacher:</strong></td>
</tr>
<tr>
<td>◊ Continuously questions students about the reasonableness of their intermediate results.</td>
</tr>
</tbody>
</table>

### Needs Improvement

<table>
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<td>◊ Identifies individual students’ efficient approaches, but does not expand understanding to the rest of the class.</td>
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<td>◊ Demonstrates the same algorithm to all related tasks although there may be other more effective approaches.</td>
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### Emerging

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<td>◊ Continuously questions students about the reasonableness of their intermediate results.</td>
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### Proficient

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<td><strong>Task:</strong></td>
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<td><strong>Teacher:</strong></td>
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<tr>
<td>◊ Continuously questions students about the reasonableness of their intermediate results.</td>
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Standards for Mathematical Practice:
Standard 8. Look For and Express Regularity in Repeated Reasoning

The Standard:
Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation \((y - 2)/(x - 1) = 3\). Noticing the regularity in the way terms cancel when expanding \((x - 1)(x + 1), (x - 1)(x^2 + x + 1), \text{ and } (x - 1)(x^3 + x^2 + x + 1)\) might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Classroom Observations:
Integrating Standard Eight into classroom practice is not only a matter of planning for lessons that occasion students to look for general methods and shortcuts. It also requires teachers to attend to and listen closely to their students’ noticings and “a-ha moments,” and to follow those a-ha moments so that they generalize to the classroom as a whole. The video clips at Inside Mathematics website: http://www.insidemathematics.org/index.php/mathematical-practice-standards are intended to show as a composite how teachers create the conditions for students to look for and express regularity in repeated reasoning, and follow and elaborate students’ thinking when they begin to make these connections.

Students:

- Look for methods and shortcuts in patterns and repeated calculations
- Evaluate the reasonableness of results and solutions

Because Teachers:

- Provide tasks and problems with patterns
- Ask about answers before and reasonableness after computations

Students might think or do:

- “When I divide 15 by 9, the 9 keeps ‘going in’ 6 times...over and over again. That means I have a repeating decimal.”
- “I solve this problem using 8 adults. Then I solved it using 10 adults, 12 adults, and 20 adults. What’s the same about my solution steps each time? How can that help me describe a process or an equation for the problem?”
- “Wait, where am I going with this? What does 5.76 represent? Where am I in the process of solving this problem?”
- “Wait, I can’t just write 7.2 because you can’t have 7.2 children in a group.”

Math Solutions

Education Development Center, Inc.
<table>
<thead>
<tr>
<th>Look For and Express Regularity in Repeated Reasoning</th>
<th>Needs Improvement</th>
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<th>Proficient</th>
<th>Exemplary</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Task:</strong></td>
<td>◊ Is disconnected from prior and future concepts.</td>
<td>◊ Is overly repetitive or has gaps that do not allow for development of a pattern.</td>
<td>◊ Reviews prior knowledge and requires cumulative understanding.</td>
<td>◊ Addresses and connects to prior knowledge in a non-routine way.</td>
</tr>
<tr>
<td><strong>Teacher:</strong></td>
<td>◊ Does not show evidence of understanding the hierarchy within concepts.</td>
<td>◊ Hides or does not draw connections to prior or future concepts.</td>
<td>◊ Lends itself to developing a pattern or structure.</td>
<td>◊ Requires recognition of pattern or structure to be completed.</td>
</tr>
<tr>
<td><strong>Teacher:</strong></td>
<td>◊ Presents or examines task in isolation.</td>
<td></td>
<td>◊ Connects concept to prior and future concepts to help students develop an understanding of procedural shortcuts.</td>
<td>◊ Encourages students to connect task to prior concepts and tasks.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>◊ Demonstrates connections between tasks.</td>
<td>◊ Demonstrates connections between tasks.</td>
<td>◊ Prompts students to generate exploratory questions based on current task.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>◊ Encourages students to monitor each other’s intermediate results.</td>
</tr>
</tbody>
</table>

**Questions to Develop Mathematical Thinking**

- Will the same strategy work in other situations?
- Is this always true, sometimes true or never true?
- How would we prove that...?
- What do you notice about...?
- What is happening in this situation?
- What would happen if...?
- Is there a mathematical rule for...?
- What predictions or generalizations can this pattern support?
- What mathematical consistencies do you notice?


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